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Dynamic Analysis of a Spinning Timoshenko Beam by the Differential Quadrature Method

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The dynamic characteristics of a spinning Timoshenko beam are investigated by using the differential quadrature method (DQM). The beam is subject to any combination of free, simply supported, clamped, and elastically supported boundary conditions. The weighting matrices in the differential quadrature formulation are modified to incorporate classical boundary conditions of the beam, whereas boundary conditions of elastic supports are incorporated into the discretized equations. Numerical results of the spinning Timoshenko beam obtained by the DQM are compared with the exact solutions or results obtained by the finite element method. The results show the high accuracy and efficiency of the differential quadrature method.

Nomenclature

A	= cross-sectional area of beam
$[C^b]$	= damping matrix of elastic supports
C_{xx}, C_{yy}	= damping coefficients
E	= Young's modulus
G	= shear modulus
$[G]$	= gyroscopic matrix
I	= transverse moment of inertia
J_p	= polar mass moment of inertia
$[K]$	= stiffness matrix
$[K^b]$	= stiffness matrix of elastic supports
K_{xx}, K_{yy}	= spring constants
l	= length of beam
$[M]$	= mass matrix
M_x, M_y	= bending moments
N	= number of sampling points
R	= radius of circular cross section of beam
u_x, u_y	= transverse deflections
V_x, V_y	= shear forces
W_{ij}	= weighting coefficients
x_i	= location of ith sampling points
κ	= shear coefficient
ρ	= mass density
φ_x, φ_y	= rotational displacements
Ω	= spinning speed of beam

Introduction

SPINNING structures can be found in rotating machinery systems such as motors, engines, and turbines. Natural frequencies and mode shapes of such structures are important in the design of systems. Flexural vibrations of beams have been of considerable interest for engineers. It is known that the classical Bernoulli–Euler beam theory predicts with adequate engineering accuracy the frequencies of flexural vibration of the lower modes of relatively long, slender beams. For beams having cross-sectional dimensions of the same order of magnitude as their lengths, or when higher modes of beams are desired, results obtained by using the Timoshenko beam theory have been shown in good agreement with results by the classical formulation of elasticity and give a good approximation to experimental results.

The response of a rotating shaft with a Timoshenko beam model was analyzed by using an integral transformation method by Katz et al.[1] The free flexural vibration of a spinning Timoshenko beam with classical boundary conditions was analytically solved by Zu and Han.[2] They also obtained the dynamic response of a spinning Timoshenko beam with general boundary conditions and subjected to a moving load.[3]

The differential quadrature method (DQM) is a numerical method that is suitable for solving initial- and/or boundary-value problems and is able to provide highly accurate results with small computational effort. This method was developed by Bellman and Casti[4] and Bellman et al.[5] in the 1970s for solving partial differential equations. The authors of Refs. 6–8 applied this method to structural problems involving fourth-order partial differential equations. Sherbourne and Pandey[9] analyzed buckling of beams and composite plates by using the DQM. Using the DQM, Laura and Gutierrez[10] studied vibration behavior of Timoshenko beams. Other investigators applied this method to other areas of applications: Shu and Richards[11] solved the two-dimensional incompressible Navier–Stokes equation by using a generalized differential quadrature method; Gutierrez and Laura[12] solved the Helmholtz equation in a parallelogram domain with mixed boundary conditions; Du et al.[13] used the generalized DQM for structural analysis. Bert and Malik[14] reviewed the recent development of the DQM in computational mechanics. Malik and Bert[15] developed three-dimensional elasticity solutions for free vibrations of rectangular plates by DQM.

However, there is a major drawback for DQM when dealing with governing equations of fourth or higher orders for which two or more boundary conditions are specified at each boundary points. Numerical error is induced by using the direct deletion or δ-point method in the original DQM because boundary conditions are not exactly satisfied at the boundary points, especially for natural boundary conditions. Choi and Chou[16] developed the modified DQM, in which modified weighting matrices are used, for structural analysis, and the just-mentioned drawback in the original DQM was overcome.

In this paper the dynamic characteristics of a spinning Timoshenko beam are investigated by using the differential quadrature method, and a new approach for application of the boundary conditions is presented. Natural frequencies of a spinning Timoshenko beam with classical boundary conditions are obtained by DQM and are compared with the exact values obtained by Zu and Han.[2] For elastically supported spinning Timoshenko beams, results obtained by the differential quadrature method are compared with those obtained by the finite element method. Excellent agreements are observed from these comparisons.

DQM

The basic idea of the differential quadrature method is that the derivative of a function, with respect to a space variable at a given sampling point, is approximated as a weighted linear sum of the

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functional values at all of the sampling points in the domain of that variable. By applying the differential quadrature (DQ) formulation, the partial differential equation is then reduced to a set of algebraic equations for time-independent problems and a set of ordinary differential equations in time for initial/boundary-value problems. As for any polynomial approach, the accuracy of the solution by this method increases as the order of the polynomial increases. Possible oscillations of numerical results arising from higher-order polynomials can be avoided by using numerical interpolation methods.

For a function $f(x)$ a DQ approximation for the first derivative at the ith sampling point is given by

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x_i) \cong \sum_{j=1}^{N} W_{ij} f(x_j), \qquad i = 1, 2, \ldots, N \tag{1}$$

where $f(x_i)$ is the functional value at the sampling point x_i. Equation (1) is the DQ rule. To determine the weighting coefficients, the function $f(x)$ is represented by a test function, such as a polynomial:

$$f(x) = x^{k-1}, \qquad k = 1, 2, \ldots, N \tag{2}$$

Substituting Eq. (2) into Eq. (1), one obtains

$$\sum_{j=1}^{N} W_{ij} x_j^{k-1} \cong (k-1) x_i^{(k-2)}, \qquad i, k = 1, 2, \ldots, N \tag{3}$$

This expression represents N sets of N linear algebraic equations for the weighting coefficients W_{ij}. The sets have a unique solution for W_{ij} because the matrix of elements x_j^{k-1} represents a Vandermonde matrix whose inverse always exists.[17] For obtaining good accuracy of the analyzed results, it is important to choose appropriate sampling points in the domain. Following Bert and Malik,[14] the sampling points are generated by the following equation:

$$x_i = \{1 - \cos[(i-1)\pi/(N-1)]\}/2, \qquad i = 1, 2, \ldots, N \tag{4}$$

Other sampling points, such as uniformly spaced ones, could be used as well. Experience shows that numerical results are more accurate by using the sampling points generated from Eq. (4) than those by using other sampling points. Once the sampling points are selected, the weighting coefficients W_{ij} can be obtained.

Equations of Motion

Figure 1 shows a uniform spinning Timoshenko beam in an inertial coordinate system $oxyz$. The beam is assumed to have a circular cross section with radius R and is supported by spring and damper sets in the x direction at both ends. For clarity, similar spring and damper sets supporting both ends of the beam in the y direction are not shown. The equations of motion for the beam in the x–z plane are[1]

$$V_x = \frac{\kappa AG}{l}\left(l\varphi_x - \frac{\partial u_x}{\partial \zeta}\right) \tag{5}$$

$$\frac{1}{l}\frac{\partial V_x}{\partial \zeta} + \rho A \frac{\partial^2 u_x}{\partial t^2} = 0 \tag{6}$$

$$M_y = \frac{EI}{l}\frac{\partial \varphi_x}{\partial \zeta} \tag{7}$$

$$-\frac{1}{l}\frac{\partial M_y}{\partial \zeta} + V_x + \rho I \frac{\partial^2 \varphi_x}{\partial t^2} + \Omega J_p \frac{\partial \varphi_y}{\partial t} = 0 \tag{8}$$

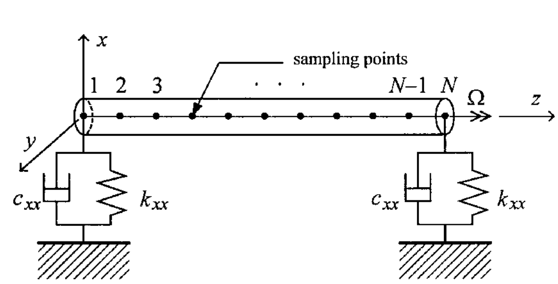

Fig. 1 Spinning Timoshenko beam with elastic supports.

and in the y–z plane are

$$V_y = \frac{\kappa AG}{l}\left(l\varphi_y - \frac{\partial u_y}{\partial \zeta}\right) \tag{9}$$

$$\frac{1}{l}\frac{\partial V_y}{\partial \zeta} + \rho A \frac{\partial^2 u_y}{\partial t^2} = 0 \tag{10}$$

$$M_x = \frac{EI}{l}\frac{\partial \varphi_y}{\partial \zeta} \tag{11}$$

$$-\frac{1}{l}\frac{\partial M_x}{\partial \zeta} + V_y + \rho I \frac{\partial^2 \varphi_y}{\partial t^2} - \Omega J_p \frac{\partial \varphi_x}{\partial t} = 0 \tag{12}$$

where $\zeta = z/l$.

By using the DQ rule, Eqs. (5) and (6) can be written in the following discrete form, respectively:

$$\{V_{xi}\} = (\kappa AG/l)\big(l\{\varphi_{xi}\} - \big[W_{ij}^{u_x}\big]\{u_{xi}\}\big) \tag{13}$$

$$\frac{1}{l}\big[W_{ij}^{V_x}\big]\{V_{xi}\} + \rho A\left\{\frac{\partial^2 u_{xi}}{\partial t^2}\right\} = 0 \tag{14}$$

where the upper indices u_x and V_x indicate that the associated weighting matrices represent differentiation of u_x and V_x, respectively, with respect to the space variable ζ. Substitution of Eq. (13) into Eq. (14) yields

$$\rho A l^2 \left\{\frac{\partial^2 u_{xi}}{\partial t^2}\right\} + \big[W_{ij}^{V_x}\big]\kappa AGl\{\varphi_{xi}\} - \big[W_{ij}^{V_x}\big]\kappa AG\big[W_{ij}^{u_x}\big]\{u_{xi}\} = 0 \tag{15}$$

Following the same procedure, the governing equations (7–12) are written in discrete form as

$$\rho Il\left\{\frac{\partial^2 \varphi_{xi}}{\partial t^2}\right\} + \Omega J_p l \left\{\frac{\partial \varphi_{yi}}{\partial t}\right\} - \big[W_{ij}^{M_x}\big]\frac{EI}{l}\big[W_{ij}^{\varphi_x}\big]\{\varphi_{xi}\} + \kappa AGl\{\varphi_{xi}\} - \kappa AG\big[W_{ij}^{u_x}\big]\{u_{xi}\} = 0 \tag{16}$$

$$\rho A l^2 \left\{\frac{\partial^2 u_{yi}}{\partial t^2}\right\} + \big[W_{ij}^{V_y}\big]\kappa AGl\{\varphi_{yi}\} - \big[W_{ij}^{V_y}\big]\kappa AG\big[W_{ij}^{u_y}\big]\{u_{yi}\} = 0 \tag{17}$$

$$\rho Il\left\{\frac{\partial^2 \varphi_{yi}}{\partial t^2}\right\} - \Omega J_p l \left\{\frac{\partial \varphi_{xi}}{\partial t}\right\} - \big[W_{ij}^{M_y}\big]\frac{EI}{l}\big[W_{ij}^{\varphi_y}\big]\{\varphi_{yi}\} + \kappa AGl\{\varphi_{yi}\} - \kappa AG\big[W_{ij}^{u_y}\big]\{u_{yi}\} = 0 \tag{18}$$

In Eqs. (16–18) the upper indices M_x, M_y, V_y, φ_x, and φ_y of the weighting matrices indicate that the associated weighting matrices represent differentiation of M_x, M_y, V_y, φ_x, and φ_y, respectively, with respect to the spatial variable ζ. Before boundary conditions are incorporated, all of the weighting matrices in Eqs. (15–18) are identical, regardless of their indices.

Equations (15–18) can be written in matrix form as

$$[M]\{\ddot{q}\} + [G]\{\dot{q}\} + [K]\{q\} = \{0\} \tag{19}$$

where

$$\{q\}_{4N\times 1} = \big\{\{u_{xi}\}^T \quad \{\varphi_{xi}\}^T \quad \{u_{yi}\}^T \quad \{\varphi_{yi}\}^T\big\}^T \tag{20}$$

$$[M]_{4N\times 4N} = \begin{bmatrix} [M_A] & [0] & [0] & [0] \\ [0] & [M_I] & [0] & [0] \\ [0] & [0] & [M_A] & [0] \\ [0] & [0] & [0] & [M_I] \end{bmatrix} \tag{21}$$

$$[G]_{4N\times 4N} = \begin{bmatrix} [0] & [0] & [0] & [0] \\ [0] & [0] & [0] & \mathrm{diag}[\Omega J_p l] \\ [0] & [0] & [0] & [0] \\ [0] & -\mathrm{diag}[\Omega J_p l] & [0] & [0] \end{bmatrix} \tag{22}$$

$$[K]_{4N\times 4N}=\begin{bmatrix}[K]_{11} & [K]_{12} & [0] & [0]\\ [K]_{21} & [K]_{22} & [0] & [0]\\ [0] & [0] & [K]_{33} & [K]_{34}\\ [0] & [0] & [K]_{43} & [K]_{44}\end{bmatrix}$$

$$[M_A]=\mathrm{diag}[\rho A l^2], \qquad [M_I]=\mathrm{diag}[\rho I l] \tag{23}$$

$[M]$ is a diagonal matrix, and $[G]$ is a skew-symmetric matrix, whereas the matrix $[K]$ contains all of the weighting matrices in Eqs. (15–18), and its elements are listed in the Appendix.

In this paper Timoshenko beams with uniform cross sections are considered. However, beams with nonuniform cross sections, such as tapered or stepped beams, could also be treated by the present approach with a slight modification to the formulation just shown.

Application of Boundary Conditions

For a Timoshenko beam the boundary conditions in the x–z plane are usually of the following types: 1) hinged end, $u_x=0$ and $M_y=0$; 2) clamped end, $u_x=0$ and $\varphi_x=0$; 3) free end, $M_y=0$ and $V_x=0$; and 4) elastically supported end, $M_y=0$ and

$$\kappa AG\frac{u_x'(\zeta_0,t)}{l}-\kappa AG\varphi_x(\zeta_0,t)=a\left(K_{xx}u_x(\zeta_0,t)+C_{xx}\frac{\partial u_x(\zeta_0,t)}{\partial t}\right) \tag{24}$$

where $a=1$ for the left end of the beam, $\zeta_0=0$, and $a=-1$ for the right end, $\zeta_0=1$.

In this paper the approach for applying boundary conditions is to modify the weighting matrices in Eq. (19) according to the end conditions of the beam. As an example, for a beam having one end clamped and the other end free in the x–z plane, the weighting matrices in Eqs. (15–18) are modified as follows:

1) At the left end $\zeta=0$: $u_x=0$ and $\varphi_x=0$.

Because u_x and φ_x are equal to zero at the first sampling point, i.e., u_{x1} and φ_{x1} are always zero, $[W_{ij}^{u_x}]$ and $[W_{ij}^{\varphi_x}]$ are then modified as

$$[\overline{W_{ij}^{u_x}}]=[\overline{W_{ij}^{\varphi_x}}]=\begin{bmatrix}0 & W_{12} & \cdots & W_{1,N-1} & W_{1N}\\ 0 & W_{22} & \cdots & W_{2,N-1} & W_{2N}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & W_{N-1,2} & \cdots & W_{N-1,N-1} & W_{N-1,N}\\ 0 & W_{N2} & \cdots & W_{N,N-1} & W_{NN}\end{bmatrix}$$

2) At the right end $\zeta=1$: $M_y=0$ and $V_x=0$.

Because M_y and V_x are equal to zero at the last sampling point, i.e., M_{yN} and V_{xN} are always zero, $[W_{ij}^{M_y}]$ and $[W_{ij}^{V_x}]$ are then modified as

$$[\overline{W_{ij}^{V_x}}]=[\overline{W_{ij}^{M_y}}]=\begin{bmatrix}W_{11} & W_{12} & \cdots & W_{1,N-1} & 0\\ W_{21} & W_{22} & \cdots & W_{2,N-1} & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ W_{N-1,1} & W_{N-1,2} & \cdots & W_{N-1,N-1} & 0\\ W_{N1} & W_{N2} & \cdots & W_{N,N-1} & 0\end{bmatrix}$$

Following the same procedure, any combination of free, simply supported, and clamped boundary conditions of a Timoshenko beam can be treated when using the DQ method.

However, for an elastically supported beam the procedure already described for applying boundary conditions is not suitable because the conditions for the shear force are now from force equilibrium. For the elastically supported beam shown in Fig. 1, boundary conditions are bending moments $M_x=0$ and $M_y=0$ at both ends, and the weighting matrices $[W_{ij}^{M_x}]$ and $[W_{ij}^{M_y}]$ in Eqs. (16) and (18) are then modified as

$$[\overline{W_{ij}^{M_x}}]=[\overline{W_{ij}^{M_y}}]=\begin{bmatrix}0 & W_{12} & \cdots & W_{1,N-1} & 0\\ 0 & W_{22} & \cdots & W_{2,N-1} & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & W_{N-1,2} & \cdots & W_{N-1,N-1} & 0\\ 0 & W_{N2} & \cdots & W_{N,N-1} & 0\end{bmatrix} \tag{25}$$

The other two boundary conditions of elastically supported ends in the x–z plane, which are given as Eq. (24), are obtained from force equilibrium. Equations (15) and (17) are also obtained from the force equilibrium equations (5) and (9). By imposing Eq. (24) on Eq. (15), the following equation is obtained:

$$\begin{bmatrix}0 & 0 & \cdots & 0 & 0\\ 0 & \rho A l^2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \rho A l^2 & 0\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix}\begin{Bmatrix}\ddot u_{x1}\\ \ddot u_{x2}\\ \vdots\\ \ddot u_{xN-1}\\ \ddot u_{xN}\end{Bmatrix}+\kappa AGl\begin{bmatrix}-1 & 0 & \cdots & 0 & 0\\ W_{21} & W_{22} & \cdots & W_{2,N-1} & W_{2N}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ W_{N-1,1} & W_{N-1,2} & \cdots & W_{N-1,N-1} & W_{N-1,N}\\ 0 & 0 & \cdots & 0 & -1\end{bmatrix}\begin{Bmatrix}\varphi_{x1}\\ \varphi_{x2}\\ \vdots\\ \varphi_{xN-1}\\ \varphi_{xN}\end{Bmatrix}$$

$$-\kappa AG\begin{bmatrix}-1 & 0 & \cdots & 0 & 0\\ W_{21} & W_{22} & \cdots & W_{2,N-1} & W_{2N}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ W_{N-1,1} & W_{N-1,2} & \cdots & W_{N-1,N-1} & W_{N-1,N}\\ 0 & 0 & \cdots & 0 & -1\end{bmatrix}\begin{Bmatrix}u'_{x1}\\ u'_{x2}\\ \vdots\\ u'_{xN-1}\\ u'_{xN}\end{Bmatrix}=l\begin{bmatrix}K_{xx} & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & -K_{xx}\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\\ \vdots\\ u_{xN-1}\\ u_{xN}\end{Bmatrix}$$

$$+l\begin{bmatrix}C_{xx} & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & -C_{xx}\end{bmatrix}\begin{Bmatrix}\dot u_{x1}\\ \dot u_{x2}\\ \vdots\\ \dot u_{xN-1}\\ \dot u_{xN}\end{Bmatrix} \tag{26}$$

where $\{u'_x\} = [W_{ij}^{u_x}]\{u_x\}$, as indicated by Eq. (1). Equation (26) represents N linear ordinary differential equations. The two equations from the first and the last rows of Eq. (26) satisfy the boundary condition (24) for the left and right ends, respectively. Equation (17) is treated in a similar manner for incorporating boundary conditions of elastic supports in the y–z plane. The system equation of motion is now written as

$$[M^*]\{\ddot{q}\} + ([G] - l[C^b])\{\dot{q}\} + ([K] - l[K^b])\{q\} = \{0\} \qquad (27)$$

where $[K^b]$ and $[C^b]$ are given in the Appendix; the matrix $[M^*]$ is the same as the matrix $[M]$ in Eq. (21) except that the first and last elements of $[M_A]$ are set to zeros.

Numerical Results

Natural frequencies of a spinning Timoshenko beam with classical boundary conditions or elastic supports are obtained by using the DQM. Geometric and material properties of the beam are listed in Table 1. The accuracy and convergence properties of the solutions obtained by using the DQM are studied. Table 2 presents the numerical results by the DQM using different numbers of sampling points. Excellent accuracy and good convergence of results are obtained. Results are sufficiently accurate by using 13 sampling points. For simply supported beams there are actually four roots for each normal mode. However, the frequencies of the second pair are of very large magnitude, and they have no physical meanings.[2]

By using the DQM with 13 sampling points, the natural frequencies of the beam with different boundary conditions and at a spinning speed of $\Omega = 500$ rad/s are obtained and presented in Table 3. The exact values presented by Zu and Han[2] are also listed in this table for comparison. All results obtained by the DQM are in excellent agreement with the exact values, and the errors are less than 0.1%. From Table 3 one can see the superb accuracy of DQM when it is applied to a spinning Timoshenko beam with classical boundary conditions.

For an elastically supported Timoshenko beam, as shown in Fig. 1, the springs and dampers are assumed to be isotropic, and their coefficients are listed in Table 1. The whirl speeds of the spinning Timoshenko beam with elastic supports are obtained by the DQM and are shown in Fig. 2. Also shown in Fig. 2 are results obtained by the finite element method (FEM) using 12 elements. For this case the same number of degrees of freedom is used for both DQM and FEM. It is observed that all whirl speeds obtained by DQM are less than the corresponding ones obtained by FEM but they have small difference (about 1%).

A parametric study is performed for the influence of support stiffness on the natural frequencies of the beam. With damping neglected the variations of the first three natural frequencies of a nonspinning beam vs the spring stiffness are shown in Fig. 3. As expected, when the spring coefficients increases from very low to very high values, the behavior of the beam is changed from that of a free–free beam to that of a hinged–hinged beam. The first three natural frequencies of the hinged–hinged beam are 734, 2876, and 6267 rad/s, and the corresponding exact values of a Bernoulli–Euler beam are 739, 2957, and 6654 rad/s.[18] The Bernoulli–Euler beam theory gives

Table 1 Geometric and material properties of the spinning Timoshenko beam

Parameter	Value
ρ	7800 kg/m^3
E	2.0×10^{11} N/m^2
G	7.7×10^{10} N/m^2
R	0.05 m
κ	0.88
l	1.3 m
K_{xx}, K_{yy}	5.0×10^7 N/m
C_{xx}, C_{yy}	5.0×10^2 N-s/m

Table 2 Natural frequencies of the spinning simply supported Timoshenko beam at $\Omega = 500$ rad/s

Number of sampling points	Mode 1, rad/s		Mode 2, rad/s		Mode 3, rad/s		Mode 4, rad/s	
	B	F	B	F	B	F	B	F
Exact[a]	732.258	735.818	2869.54	2882.80	6254.04	6280.74	10676.8	10718.1
DQM, $N = 9$	732.258	735.818	2869.67	2882.93	6248.40	6275.08	10934.6	10976.6
DQM, $N = 10$	732.258	735.818	2869.54	2882.79	6255.60	6282.31	10627.6	10668.8
DQM, $N = 11$	732.258	735.818	2869.54	2882.80	6253.88	6280.58	10687.4	10728.8
DQM, $N = 12$	732.258	735.818	2869.54	2882.80	6254.01	6280.71	10675.2	10716.5
DQM, $N = 13$	732.258	735.818	2869.54	2882.80	6254.04	6280.74	10676.6	10717.9
DQM, $N = 14$	732.258	735.818	2869.54	2882.80	6254.04	6280.74	10676.8	10718.1
DQM, $N = 15$	732.258	735.818	2869.54	2882.80	6254.04	6280.74	10676.8	10718.1

[a]Reference 2; B: backward, F: forward.

Table 3 Natural frequencies of the spinning Timoshenko beam at $\Omega = 500$ rad/s

Boundary conditions		Mode 1, rad/s		Mode 2, rad/s		Mode 3, rad/s		Mode 4, rad/s	
		B	F	B	F	B	F	B	F
Hinged–hinged	DQM	732.258	735.818	2869.54	2882.80	6254.04	6280.74	10676.6	10717.9
	Exact[a]	732.258	735.818	2869.54	2882.80	6254.04	6280.74	10676.8	10718.1
	Error	0%	0%	0%	0%	0%	0%	0%	0%
Clamped–clamped	DQM	1626.94	1631.02	4329.44	4343.24	8136.95	8163.07	12809.1	12847.4
	Exact[a]	1626.94	1631.02	4329.45	4343.25	8132.10	8157.92	12803.9	12842.3
	Error	0%	0%	0%	0%	0.06%	0.07%	0.04%	0.04%
Free–free	DQM	0	0	0	0	1641.36	1658.89	4408.59	4444.36
	Exact[a]	0	0	0	0	1641.37	1658.89	4408.60	4444.37
	Error	0%	0%	0%	0%	0%	0%	0%	0%
Hinged–free	DQM	0	0	1137.25	1147.01	3602.57	3626.27	7265.29	7305.53
	Exact[a]	0	0	1137.25	1147.01	3602.57	3626.27	7265.40	7305.65
	Error	0%	0%	0%	0%	0%	0%	0%	0%
Clamped–free	DQM	261.635	263.334	1607.18	1618.49	4370.97	4395.50	8232.56	8272.69
	Exact[a]	261.635	263.334	1607.18	1618.49	4370.98	4395.52	8232.94	8273.07
	Error	0%	0%	0%	0%	0%	0%	0%	0%
Clamped–hinged	DQM	1133.80	1137.83	3573.51	3587.19	7179.96	7206.41	11741.4	11781.4
	Exact[a]	1133.80	1137.83	3573.51	3587.19	7180.06	7206.51	11739.2	11779.2
	Error	0%	0%	0%	0%	0%	0%	0.02%	0.02%

[a]Reference 2; B: backward, F: forward.

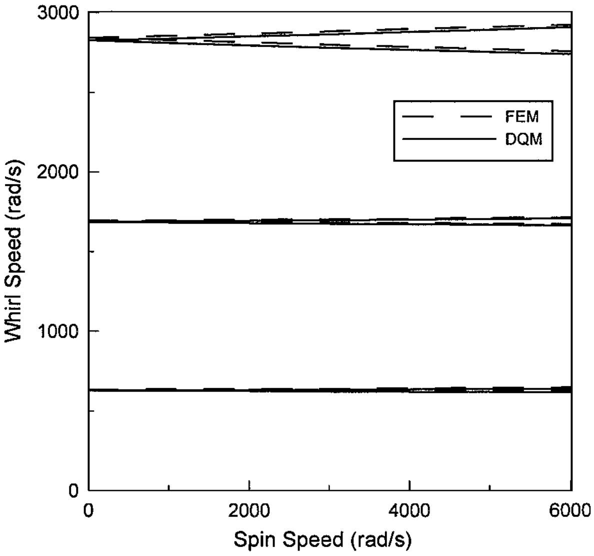

Fig. 2 Whirl speed map of the spinning Timoshenko beam.

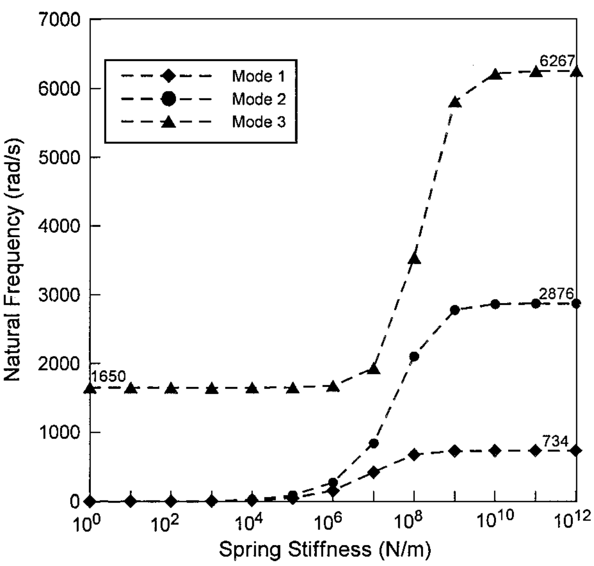

Fig. 3 Natural frequencies of Timoshenko beam with different spring coefficients.

higher values of natural frequencies than the Timoshenko beam theory does, especially for higher modes.

Conclusions

In this paper the dynamic characteristics of a spinning Timoshenko beam are studied by using the DQM. Boundary conditions of elastic supports are incorporated into the discretized equations. Numerical results of whirl speeds for beam with classical or elastic-support boundary conditions are obtained. Excellent agreements are observed from comparison of the DQM results and exact solutions or results obtained by the FEM. The DQM is an accurate and efficient numerical method for dynamic analysis of elastically supported spinning beams.

Appendix: Matrices [K], [C^b], and [K^b]

Elements of the matrix $[K]$ in Eq. (23) are

$$[K]_{11} = -\left[W_{ij}^{V_x}\right]\kappa AG\left[W_{ij}^{u_x}\right], \qquad [K]_{12} = -\kappa AGl\left[W_{ij}^{\varphi_x}\right]$$

$$[K]_{21} = -\kappa AG\left[W_{ij}^{u_x}\right]$$

$$[K]_{22} = \mathrm{diag}[\kappa AGl] - \left[W_{ij}^{M_x}\right](EI/l)\left[W_{ij}^{\varphi_x}\right]$$

$$[K]_{33} = -\left[W_{ij}^{V_y}\right]\kappa AG\left[W_{ij}^{u_y}\right], \qquad [K]_{34} = -\kappa AGl\left[W_{ij}^{\varphi_y}\right]$$

$$[K]_{43} = -\kappa AG\left[W_{ij}^{u_y}\right]$$

$$[K]_{44} = \mathrm{diag}[\kappa AGl] - \left[W_{ij}^{M_y}\right](EI/l)\left[W_{ij}^{\varphi_y}\right]$$

Matrices $[C^b]$ and $[K^b]$ in Eq. (27) are

$$[C^b]_{4N\times 4N} = \begin{bmatrix} [C^b_{xx}] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] \\ [0] & [0] & [C^b_{yy}] & [0] \\ [0] & [0] & [0] & [0] \end{bmatrix}$$

$$[K^b]_{4N\times 4N} = \begin{bmatrix} [K^b_{xx}] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] \\ [0] & [0] & [K^b_{yy}] & [0] \\ [0] & [0] & [0] & [0] \end{bmatrix}$$

$$\left[C^b_{ii}\right]_{N\times N} = \begin{bmatrix} C_{ii} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -C_{ii} \end{bmatrix}, \qquad i = x, y$$

$$\left[K^b_{ii}\right]_{N\times N} = \begin{bmatrix} K_{ii} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -K_{ii} \end{bmatrix}, \qquad i = x, y$$

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